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Translated by M.D.F.

PMM. U.S.S.R., Vol.49, No.4, pp., 498-503, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

STABILITY OF BODIES MADE OF NON-HOMOGENEOUSLY AGING ANISOTROPIC, VISCOELASTIC MATERIAL *

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Results of the study of stability of compressed rods made of a non-homogeneously aging viscoelastic material are generalized to the case of an arbitrary body with anisotropy.

Let us consider a body acted upon by volume forces $\mathbf{F}$ and surface loads $\mathbf{q}$ applied at the boundary of the body S_q, in an orthogonal $x_i\ (i = 1, 2, 3)$, $\mathbf{F} = \{F_i\}$, $\mathbf{q} = \{q_i\}$ coordinate system. The points of the body undergo, under the action of these forces, the displacements $u_i(t, \mathbf{x})$ determining the trajectory of the unperturbed motion.

Let us assume that in the initial state the body has a small initial distortion $\alpha v_i^\circ(\mathbf{x})$. In this case the body undergoes additional displacements $\alpha v_i(t, \mathbf{x})$ so that the total displacement is $u_i^* = u_i + \alpha(v_i + v_i^\circ)$. The parameter α is introduced arbitrarily (and can be assumed equal to unity). The motion of the body determined by the displacements u_i^* will be called perturbed, and the displacements αv_i will be called perturbations.

Let us introduce the displacement norm (V is the volume of the body)

$$\|\mathbf{u}\| = \Big(\int_V u_i u_i\, dV\Big)^{1/2}.$$

Here and henceforth the repeated indices denote summation.

Definition. An unperturbed motion of a viscoelastic body will be called stable, if for any number $A > 0$ a number $\delta = \delta(A) > 0$ can be found such that for any initial distortion αv_i° satisfying the inequality $\alpha\|\mathbf{v}^\circ\| < \delta$, the corresponding displacements αv_i satisfy the inequality $\alpha\|\mathbf{v}\| < A$, $0 \leqslant t < \infty$.

If the motion of the body is studied within a finite time interval $[0, T]$ and the critical value of the displacement norm $\|\mathbf{v}\|^*$ is given, we can speak of the critical time t_*, defining it as the instant at which the displacement norm $\alpha\|\mathbf{v}\|$ first attains the value $\|\mathbf{v}\|^*$: $\alpha \max\|\mathbf{v}(t)\| < \|\mathbf{v}\|^*$, $0 \leqslant t < t_*$ with $\alpha\|\mathbf{v}(t_*)\| = \|\mathbf{v}\|^*$.

We shall call the body stable in the time interval $[0, T]$, if $t_* > T$.

Analogous definitions of stability were used in connection with the non-homogeneously aging viscoelastic rods in [1, 2] where $\sup_{t,x}|y(t, x)|$, $x \in [0, l]$ (l is the rod length) was used as the rod deflection norm.

Assuming that the deformations are small, we write the equations of state for the material in the form /1/

$$\sigma_{ij} = (E_{ijkl} - \mathbf{R}_{ijkl})\,\varepsilon_{kl} \tag{1}$$

$$E_{ijkl} = E_{ijkl}(t + \rho(\mathbf{x})),\quad \mathbf{R}_{ijkl}\varepsilon_{kl} = \int_0^t R_{ijkl}^\rho \varepsilon_{kl}(\tau)\, d\tau,\quad R_{ijkl}^\rho = R_{ijkl}(t + \rho(\mathbf{x}), \tau + \rho(\mathbf{x}))$$

The moduli of elasticity E_{ijkl} and relaxation kernels R_{ijkl}^ρ of the material satisfy the following relations:

**Prikl. Matem. Mekhan.*, 49, 4, 648-654, 1985.

$$\lim_{t\to\infty} E_{ijkl} = E^{*}_{ijkl} = \mathrm{const} \tag{2}$$

$$0 \leqslant R^{\rho}_{ijkl} \leqslant R^{*}_{ijkl}(t,\tau),\ R^{*\prime}_{ijkl} = \sup_{t>0}\int_0^t R^{*}_{ijkl}(t,\tau)\,d\tau$$

$$\int_T^t \sup_{\mathbf{x}} |R^{\rho}_{ijkl} - R^{\circ}_{ijkl}(t,\tau)|\,d\tau \to 0 \quad \text{as} \quad T\to\infty$$

$$\lim_{T\to\infty}\sup_{t\geqslant T}\int_T^t R^{\circ}_{ijkl}(t,\tau)\,d\tau = R^{\circ\prime}_{ijkl}$$

The function $\rho(\mathbf{x})$ which has continuous first derivatives in the whole region occupied by the body, determines the age of the material points with coordinates $\mathbf{x}$, at the instant of application of the external load.

Assuming that the external loads are conservative (dead weight), we shall write the functional /3/ as

$$Э = \int_V \left[\frac{1}{2} E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \varepsilon_{ij}(\mathbf{R}_{ijkl}\varepsilon_{kl})\right] dV - \int_V F_i u_i^* \,dV - \int_{S_q} q_i u_i^* \,dS$$

$$\varepsilon_{ij} = \frac{1}{2}\{(u_{i,j} + u_{j,i}) + \alpha(v_{i,j} + v_{j,i}) +$$
$$[(u_{k,i} + \alpha v_{k,i} + \alpha \overset{\circ}{v}_{k,i})(u_{k,j} + \alpha v_{k,j} + \alpha\overset{\circ}{v}_{k,j}) - \alpha^2 \overset{\circ}{v}_{k,i}\overset{\circ}{v}_{k,j}]\}$$

Let us vary the functional $Э$ over the displacements v_i at the running instant of time t (the displacements u_i corresponding to the unperturbed motion are not varied).

As we know /3/, the condition for the functional $Э$ to be stationary is, that its first variation is equal to zero

$$\delta Э = \alpha\delta Э' + \alpha^2 \delta Э'' = 0 \tag{3}$$

Here $\delta Э', \delta Э''$ are the expressions in the variation $\delta Э$ accompanying the corresponding powers of the parameter α.

We note that since the body is in equilibrium, the equation $\delta Э' = 0$ must be satisfied in the unperturbed motion. Then from (3) we obtain

$$\delta Э'' = 0 \tag{4}$$

We further assume that the displacements u_i in the unperturbed motion of the viscoelastic body are small and can be found from the equations of the linear theory of viscoelasticity. In this case we can write Eq. (4) as follows:

$$\int_V \{\delta v_{i,j}[(E_{ijkl} - \mathbf{R}_{ijkl})v_{k,l}] + \sigma_{ij}(v_{k,i} + \overset{\circ}{v}_{k,i})\,\delta v_{k,j}\}\,dV = 0 \tag{5}$$

where σ_{ij} are the stresses in the unperturbed motion of the body and δv_i are the variations in the displacements v_i. We note that (5) is equivalent to three equations of equilibrium of the body and the boundary conditions at its surface in the unperturbed motion written in terms of the perturbations.

Let us take, as the variations in the displacements δv_i, the displacements v_i themselves. Then

$$\int_V \{v_{i,j}[(E_{ijkl} - \mathbf{R}_{ijkl})v_{k,l}] + \sigma_{ij}(v_{k,i} + \overset{\circ}{v}_{k,i})v_{k,j}\}\,dV = 0\,. \tag{6}$$

We will assume that the external load acting on the body is one-parametric, i.e.

$$\sigma_{ij} = -\beta\sigma_{ij}^{\circ},\ \beta = \mathrm{const} \tag{7}$$

and such, that for any instant of time $t \geqslant 0$ the smallest eigenvalue λ_1 of the homogeneous boundary value problem

$$\int_V E_{ijkl} v_{i,j} v_{k,l}\,dV = \lambda \int_V \sigma_{ij}^{\circ} v_{k,i} v_{k,j}\,dV \tag{8}$$

is postive, i.e. $\lambda_1 \geqslant a > 0$.

Let us denote by $\mathbf{v}'$ the vector with components $v_{i,j}$ $(v' = [v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, \ldots, v_{3,3}])$. We define the scalar product of two vectors $\mathbf{v}_1', \mathbf{v}_2'$ and the norm of the vector $\mathbf{v}'$ as follows:

$$(\mathbf{v}_1'\mathbf{v}_2') = \int_V v^{(1)}_{i,j} v^{(2)}_{i,j}\,dV, \quad \|\mathbf{v}'\| = \left(\int_V v_{i,j}v_{i,j}\,dV\right)^{1/2}$$

Let us write Eq. (6) in the form

$$I = \beta I_1 + \beta I_2 + I_3 \tag{9}$$

$$I = \int_V v_{i,j} E_{ijkl} v_{k,l}\, dV, \quad I_1 = \int_V \sigma_{ij}{}^{\circ} v_{k,i} v_{k,j}\, dV$$

$$I_2 = \int_V \sigma_{ij}{}^{\circ} \overset{\circ}{v}_{k,i} v_{k,j}\, dV, \quad I_3 = \int_V v_{i,j} (\mathbf{R}_{ijkl} v_{k,l})\, dV$$

We know /4/ that the homogeneous boundary value problem described by (8) is selfconjugate and its eigenvalues are real. Then we have /5/

$$I \geqslant \lambda_1 I_1 \tag{10}$$

From (9), (10) it follows that

$$(1 - \beta/\lambda_1) I \leqslant \beta I_2 + I_3 \tag{11}$$

Using the same representations we can write, in turn,

$$I \geqslant \lambda_1{}^* \| \mathbf{v}' \|^2.$$

Thus the left-hand side of inequality (11) does not exceed the value

$$(1 - \beta/\lambda_1) \lambda_1{}^* \| \mathbf{v}' \|^2 \leqslant \beta I_2 + I_3 \tag{12}$$

We note that in general λ_1 and $\lambda_1{}^*$ are functions of time, since $E_{ijkl} = E_{ijkl}(t + \rho(\mathbf{x}))$ and $\sigma_{ij}{}^{\circ} = \sigma_{ij}{}^{\circ}(t, \mathbf{x})$. Considering the finite time interval $[0, T]$, we choose on it the minimum value (denoting it by c) of the multiplier appearing on the left-hand side of inequality (12).

Let $|\sigma|_{max}$ be the principal stress, largest in modulo, at a point of the body depending also on x and t. Then, applying the Cauchy inequality, we write the following estimate for the first term on the right-hand side of (12):

$$I_2 \leqslant \sigma \| \mathbf{v}' \| \cdot \| \mathbf{v}^{\circ\prime} \|, \quad \sigma = \sup_{\mathbf{x} \in V} |\sigma|_{max}$$

To estimate the second term in the same inequality, we use the largest eigenvalue $R_{max}(t, \tau)$ of the 9-th order matrix R^{ρ}_{ijkl}.

The relation /6/ $R_{max}(t,\tau) \leqslant \sqrt{\Sigma (R^{\rho}_{ijkl})^2}$ holds for $R_{max}(t,\tau)$. If $R^*_{ijkl}(t,\tau) \geqslant R^{\rho}_{ijkl}$, then $R_{max}(t,\tau) \leqslant \sqrt{\Sigma (R^*_{ijkl}(t,\tau))^2} = R(t,\tau)$ and we have

$$\int_V v_{i,j}(t, \mathbf{x}) \int_0^t R^{\rho}_{ijkl} v_{k,l}(\tau, \mathbf{x})\, dV\, d\tau \leqslant \| \mathbf{v}'(t) \| \int_0^t R(t,\tau) \| \mathbf{v}'(\tau) \|\, d\tau$$

As a result, we can write (12) in the form

$$c \| \mathbf{v}'(t) \| \leqslant c_1 + \int_0^t R(t,\tau) \| \mathbf{v}'(\tau) \|\, d\tau, \tag{13}$$

$$c_1 = \beta\sigma \| \mathbf{v}^{\circ\prime} \|$$

If the function $R(t,\tau)$ has no weak singularity at $t = \tau$, we can find a function $R_1(\tau)$ in the time interval $[0, T]$ such, that

$$R_1(\tau) = \sup_{t \in [0, T]} R(t, \tau) \tag{14}$$

Then (13) will yield the following inequality:

$$\| \mathbf{v}'(t) \| \leqslant \frac{c_1}{c} + \frac{1}{c} \int_0^t R_1(\tau) \| \mathbf{v}'(\tau) \|\, d\tau \tag{15}$$

If, on the other hand, the function $R(t,\tau)$ has a weak singularity at $t = \tau$, then we can handle the inequality just as was done in /2/. Thus we pass, in the inequality (13), from the kernel $R(t,\tau)$ to the iterated kernel. As we know /7/, from some number n onwards the iterated kernels become regular, and we can find the function $R_1(\tau)$ for such kernels. As a result, we pass from inequality (13) to an inequality analogous to (15).

Applying to the inequality (15) the Gronwall-Bellman lemma /8/ we obtain

$$\| \mathbf{v}'(t) \| \leqslant \frac{c_1}{c} \exp\left[\frac{1}{c} \int_0^t R_1(\tau)\, d\tau\right] \tag{16}$$

To study the stability of the body over an infinite time interval, we return to Eq. (6), rewriting it as follows:

$$\int_V v_{i,j} E^*_{ijkl} v_{k,l}\, dV - \beta K_1 = K_2 + K_3 + \beta K_4 + \beta K_5 \tag{17}$$

$$K_1 = \int_V \sigma_{ij}^* v_{k,i} v_{k,j}\, dV, \quad K_2 = \int_V v_{i,j} (E^{\circ}_{ijkl} - E_{ijkl}) v_{k,l}\, dV,$$
$$K_3 = \int_V v_{i,j} \mathbf{R}_{ijkl} v_{k,l}\, dV, \quad K_4 = \int_V (\sigma_{ij}^{\circ} - \sigma_{ij}^*) v_{k,i} v_{k,j}\, dV,$$
$$K_5 = \int_V \sigma_{ij}^{\circ} v^{\circ}_{k,i} v_{k,j}\, dV$$

In accordance with the constraints (2) imposed on the functions $E_{ijkl}(t + \rho(\mathbf{x}))$, R^{ρ}_{ijkl}, $\sigma_{ij}^{\circ}(t, \mathbf{x})$, we can find for an arbitrarily small number $A > 0$, an instant of time $T = T(A)$ such that the following inequalities will hold for every instant of time $t > T$:

$$|E^{\circ}_{ijkl} - E_{ijkl}| < A, \quad |\sigma_{ij}^{\circ} - \sigma_{ij}^*| < A$$
$$\int_T^t \sup_{\mathbf{x}} |R^{\circ}_{ijkl}(t, \tau) - R^{\rho}_{ijkl}|\, d\tau < A$$

Then we have the following relations for $t > T$:

$$K_2 < 9A\, \| \mathbf{v}'(t) \|^2, \quad K_4 < 3A\, \| \mathbf{v}'(t) \|^2 \tag{18}$$
$$K_3 = \int_V v_{i,j}(t, \mathbf{x}) \int_0^t R^{\rho}_{ijkl} v_{k,l}(\tau, \mathbf{x})\, d\tau\, dV =$$
$$\int_V v_{i,j}(t, \mathbf{x}) \Big\{ \int_0^T R^{\rho}_{ijkl} v_{k,l}(\tau, \mathbf{x})\, d\tau -$$
$$\int_T^t [R^{\rho}_{ijkl} - R^{\circ}_{ijkl}(t, \tau)]\, v_{k,l}(\tau, \mathbf{x})\, d\tau -$$
$$\int_T^t R^{\circ}_{ijkl}(t, \tau) v_{k,l}(\tau, \mathbf{x})\, d\tau \Big\}\, dV < 9A\, \| \mathbf{v}'(t) \| \cdot \| \boldsymbol{\omega}(t) \| +$$
$$9R^{*'}\, \| \mathbf{v}'(t) \| \cdot \| \boldsymbol{\omega}(T) \| + \int_V v_{i,j}(t, \mathbf{x}) \int_T^t R^{\circ}_{ijkl}(t, \tau) v_{k,l}(\tau, \mathbf{x})\, d\tau\, dV$$
$$K_5 = \int_V (\sigma_{ij}^{\circ} - \sigma_{ij}^*) v^{\circ}_{k,i} v_{k,j}\, dV + \int_V \sigma_{ij}^* v^{\circ}_{k,i} v_{k,j}\, dV <$$
$$3A\, \| \mathbf{v}'(t) \| \cdot \| \mathbf{v}^{\circ'} \| + 3\sigma^*\, \| \mathbf{v}'(t) \| \cdot \| \mathbf{v}^{\circ'} \|$$

. . . .

$$R^{*'} = \max_{i,j,k,l} R^{*'}_{ijkl}, \quad \sigma^* = \max_{\mathbf{x} \in V} \sigma^*_{\max}$$
$$\| \boldsymbol{\omega}(T) \| = \max \| \mathbf{v}'(t) \| \quad \text{when} \quad 0 \leqslant t \leqslant T$$
$$\| \boldsymbol{\omega}(t) \| = \max \| \mathbf{v}'(t) \| \quad \text{when} \quad T \leqslant t$$

where $\sigma^*_{\max}$ is the principal stress largest in modulo, at the point at which the stress state is characterized by the tensor σ_{ij}^*.

Taking into account (18) we obtain, from (17),

$$K^* < \beta K_1 + \| \mathbf{v}'(t) \| \big(12A\, \| \mathbf{v}'(t) \| + 3A\beta\, \| \mathbf{v}^{\circ'} \| + 3\sigma^*\beta\, \| \mathbf{v}^{\circ'} \| + 9A\, \| \boldsymbol{\omega}(t) \| + 9R^*\, \| \boldsymbol{\omega}(T) \| \big) + K_6 \tag{19}$$
$$K^* = \int_V v_{i,j}(t, \mathbf{x}) (E^{\circ}_{ijkl} - R^{\circ'}_{ijkl}) v_{k,l}(t, \mathbf{x})\, dV$$
$$K_6 = \int_V v_{i,j}(t, \mathbf{x}) \Big[\int_T^t R^{\circ}_{ijkl}(t, \tau) v_{k,l}(\tau, \mathbf{x})\, d\tau - R^{\circ'}_{ijkl} v_{k,l}(t, \mathbf{x}) \Big]\, dV$$

In accordance with conditions (2) we have

$$\int_T^t R^{\circ}_{ijkl}(t, \tau)\, d\tau = R^{\circ'}_{ijkl} + A$$

Then ($\delta(t - \tau)$ is the delta function)

$$\int_T^t R^{'}_{ijkl}(t, \tau) v_{k,l}(\tau, \mathbf{x})\, d\tau - R^{\circ'}_{ijkl} v_{k,l}(t, \mathbf{x}) = \tag{20}$$
$$\int_T^t [R^{\circ}_{ijkl}(t, \tau) - \delta(t - \tau)\, R^{\circ'}_{ijkl}]\, v_{k,l}(\tau, \mathbf{x})\, d\tau \leqslant A\omega_{k,l}(t, \mathbf{x})$$
$$\omega_{k,l}(t, \mathbf{x}) = \sup |v_{k,l}(t, \mathbf{x})| \quad \text{when} \quad t \geqslant T$$

The last term of relation (19) can be estimated, taking inequality (20) into account, as follows:

$$K_6 \leqslant 18\ A \| \mathbf{v}'(t) \| \cdot \| \boldsymbol{\omega}(t) \| \tag{21}$$

As a result, inequality (19) becomes

$$K^* < \beta K_1 + \| \mathbf{v}'(t) \| (12A \| \mathbf{v}'(t) \| + 27A \| \boldsymbol{\omega}(t) \| + 3A\beta \| \mathbf{v}^{\circ\prime} \| + 3\beta\sigma^* \| \mathbf{v}^{\circ\prime} \| + 9R^* \| \boldsymbol{\omega}(T) \|) \tag{22}$$

Let us consider a homogeneous boundary value problem for which we have the corresponding equation

$$K^* = \lambda K_1 \tag{23}$$

As we know /4/, the above problem is selfconjugate and its eigenvalues are real. Let us make a natural assumption as regards the symmetric matrix $\overset{\circ}{E}_{ijkl} - \overset{\circ\prime}{R}_{ijkl}$, namely, that all its eigenvalues are positive.

It can be shown that in the case when the stresses σ_{ij}^* are small, the functional $K^* - K_1$ is positive definite (excluding from our discussion the possibility of rigid displacements of the body). Then /5/ we have

$$K^* \geqslant \lambda_1 K_1 \tag{24}$$

where λ_1 is the smallest eigenvalue of the homogeneous boundary value problem (23).

As we know /5/, the following estimate holds:

$$K^* \geqslant \lambda_1^\circ \| \mathbf{v}' \|^2 \tag{25}$$

where λ_1° is the smallest eigenvalue of the homogeneous boundary value problem

$$K^* = \lambda \int_V A_{ijkl} v_{i,j} v_{k,l}\, dV, \quad A_{ijkl} = \begin{cases} 1, & i = j = k = l \\ 0 & \text{in all remaining cases.} \end{cases}$$

Thus, taking into account the estimates (24), (25), we can write inequality (22) in the form

$$(1 - \beta/\lambda_1)\, \lambda_1^\circ \| \mathbf{v}'(t) \|^2 < \| \mathbf{v}'(t) \| (12A \| \mathbf{v}'(t) \| + 18A \| \boldsymbol{\omega}(t) \| + 3A\beta \| \mathbf{v}^{\circ\prime} \| + 3\sigma^*\beta \| \mathbf{v}^{\circ\prime} \| + 9R^* \| \boldsymbol{\omega}(T) \|) \tag{26}$$

from which follows

$$(\lambda_1 - \beta - 30A\lambda_1/\lambda_1^\circ) \| \boldsymbol{\omega}(t) \| < 3\, [(A + \sigma^*)\, \beta \| \mathbf{v}^{\circ\prime} \| + 3R^* \| \boldsymbol{\omega}(T) \|],\ t > T \tag{27}$$

Assuming that the region occupied by the body is star-like relative to all points of some sphere lying within the region in question, we can write the inequality /9/

$$\| \mathbf{v}(t) \| \leqslant c^* \| \mathbf{v}'(t) \| \leqslant c^* \| \boldsymbol{\omega}(t) \| \tag{28}$$

where c^* is a constant depending only on the geometry of the body.

Thus from the inequalities (27), (28) it follows that the body in question is stable over an infinite time interval, provided that $\lambda_1 > \beta$. This implies, in particular, that when the stresses σ_{ij}° are constant with respect to time, the value of the critical time is found in the same manner as in case of an elastic body whose moduli of elasticity are replaced by the sustained moduli $\overset{\circ}{E}_{ijkl} - \overset{\circ\prime}{R}_{ijkl}$.

The stability of the body over a finite time interval can be studied using the inequality (16), (28), from which we obtain

$$\| \mathbf{v}(t) \| \leqslant J, \quad J = b \exp\left[\frac{1}{c} \int_0^t R_1(\tau)\, d\tau\right], \quad b = c^* \frac{c_1}{c}$$

The quantity which gives a lower estimate for the critical time can be found from the non-linear equation

$$J = \| \mathbf{v} \|^*$$

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Translated by L.K.

PMM.U.S.S.R.,Vol.49,No.4,pp.503-509,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

GENERALIZED SOLUTIONS OF THE DYNAMIC PROBLEM OF PERFECT ELASTOPLASTICITY *

S.B. KUKSIN

The concept of a generalized solution of an initial boundary value problem for the system of Pradtl-Reuss equations is introduced. It is shown that a generalized solution exists and is unique, and represents within the domain of elasticity a solution of the initial-boundary value problem of the dynamic theory of elasticity. An effective method for the approximate determination of the generalized solution is given, and conditions at its strong discontinuities are obtained. The basic results of this paper were published earlier without proof in /1, 2/.

1. The Prandtl-Reuss equations. Let a perfect elastoplastic body occupy a three-dimensional region Ω with a smooth boundary D. The state of the body is characterized, in Lagrange coordinates, by the stress tensor $\tau_{ij}(t, x)$, the velocity of the body particles $v_i(t, x)$, the elastic strain rate tensor $\varepsilon_{ij}(v) \equiv (v_{i,j} + v_{j,i})/2$ and the plastic strain rate tensor $\lambda_{ij}(t, x)$ ($1 \leqslant i, j \leqslant 3$, $0 \leqslant t \leqslant T, x \in \Omega$ everywhere). We assume that the measurable part D_1 of the boundary D is free and, that the displacement rate is specified on the part $D_2 = D \setminus D_1$. The density of the body is assumed constant. Assuming that it is equal to unity, we write the equations of elastoplastic flow and initial-boundary conditions /3/ thus

$$a_{ijkh}\tau_{kh} - \varepsilon_{ij}(v) + \lambda_{ij} = 0 \tag{1.1}$$

$$v_i' - \tau_{ij,j} = F_i(t, x) \tag{1.2}$$

$$(\tau_{ij}n_j)(t, x) = 0, \quad x \in D_1; \quad v_i(t, x) = v_i^\circ(t, x), \quad x \in D_2 \tag{1.3}$$

$$\tau_{ij}(0, x) = \tau_{0ij}(x), \quad v_i(0, x) = v_{0i}(x) \tag{1.4}$$

where a_{ijkh} are the coefficients of elasticity, $n_i(x), x \in D$ is the outer normal to Ω, and a prime denotes a time differential. We will supplement (1.1)—(1.4) with the von Mises condition of plasticity /3/ (τ_{ij}^D is the deviator of the tensor τ_{ij})

$$\tau_{ij}^D(t, x)\,\tau_{ij}^D(t, x) \leqslant c_*^2 \tag{1.5}$$

The equations (1.1)—(1.5) are closed by the Prandtl-Reuss relations connecting the stresses with the plastic strain rate

$$\lambda_{ij}(t, x) = \varkappa\sigma_{ij}^D(t, x), \quad \varkappa \geqslant 0$$

where $\varkappa = 0$ when inequality (1.5) is rigorously satisfied. The Prandtl-Reuss relations can be conveniently replaced by the equivalent Drucker postulate /4/. We shall write it in the integrated form

$$\int \lambda_{ij}(t, x)(\tau_{ij}(t, x) - \sigma_{ij}(t, x))\,dx \geqslant 0 \tag{1.6}$$

where σ_{ij} is a tensor field continuously differentiable in $[0, T] \times (\Omega \cup D)$, such that

$$(\sigma_{ij}^D\sigma_{ij}^D)(t, x) \leqslant c_*^2; \quad \sigma_{ij}(t, x)\,n_j(x) = 0, \quad \forall x \in D_1 \tag{1.7}$$

The initial-boundary value problem (1.1) — (1.7) was studied earlier by Duvaut and Lions, who showed in /6/ the unique solvability of the evolutionary variational inequality following from (1.1) — (1.7), satisfied by the stress tensor integrated with respect to time. Below we apply

*Prikl.Matem.Mekhan.,49,4, 655-662,1985